Role of pressure in turbulence

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There is very limited knowledge of the kinematical relations for the velocity structure functions higher than three. Instead, the dynamical equations for the structure functions of the velocity increment are obtained from the Navier Stokes equation under the assumption of the local homogeneity and isotropy. These equations contain the correlation between the velocity and pressure gradient increments, which is very difficult to know theoretically and experimentally. We have examined these dynamical relations by using direct numerical simulation data at very high resolution at large Reynolds numbers, and found that the contribution of the pressure term is important to the dynamics of the longitudinal velocity with large amplitudes. The pressure term is examined from the view point of the conditional average and the role of the pressure term in the turbulence dynamics is discussed.

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INTRODUCTION

In Kolmogorov's theory (1941, hereafter K41), the moments of velocity increments in the isotropic turbulence scale as $\langle U^p\rangle \propto r^{\zeta_p}$ with $\zeta_p = p/3$, where U is the longitudinal increment defined as $U = (\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x}))\cdot \boldsymbol{r}/r$[1, 2]. However, it is now widely accepted from the experimental and direct numerical simulation (DNS) data that at high Reynolds numbers the intermittency builds up with decrease of scales and scaling exponent ζ_p does not obey K41 and is a non-decreasing function of p.

Arguments have recently been raised as to whether or not the scaling exponent of the longitudinal structure function is equal to that of the transverse one at the order higher than three [3–9]. The structure function (moment) of the velocity increment is defined by

$$S_{m,n}(r) = \langle U^m V^n\rangle, \tag{1}$$

where V is the transverse velocity increment. In the inertial range at very high Reynolds numbers, $S_{m,n}(r)$ is assumed to obey power law $S_{m,n}(r) \propto r^{\zeta_{m,n}}$. In the homogeneous isotropic turbulence, the incompressible condition yields the well known relations at the second and third order

$$S_{0,2}(r) = S_{2,0}(r) + \frac{r}{2}\frac{d}{dr}S_{2,0}, \tag{2}$$

$$S_{1,2}(r) = \frac{1}{6}S_{3,0}(r) + \frac{r}{6}\frac{d}{dr}S_{3,0}(r), \tag{3}$$

from which it follows $\zeta_{2,0} = \zeta_{0,2}$ and $\zeta_{3,0} = \zeta_{1,2}$. Kolmogorov's 4/5 and 4/3 laws are derived from the Navier-Stokes equation as

$$S_{3,0}(r) = -\frac{4}{5}\bar{\epsilon}r, \quad S_{3,0}(r) + 2S_{1,2}(r) = -\frac{4}{3}\bar{\epsilon}r, \tag{4}$$

so that $\zeta_{3,0} = \zeta_{1,2} = 1$, where $\bar{\epsilon}$ is the average rate of energy dissipation rate per unit mass. However, for the order higher than three, no equations are known that tie directly the longitudinal structure functions to the transverse or mixed structure functions. Instead, relations among various type of structure functions are derived from the Navier-Stokes equation which contains the pressure and viscous terms[14, 18, 20] such as

$$\left\langle(\delta_r(\nabla_\parallel p))U^pV^q\right\rangle, \quad \left\langle(\delta_r(\nabla^2 U))U^pV^q\right\rangle \tag{5}$$

where

$$\delta_r(\nabla_\parallel p) = \left(\frac{\partial p(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1} - \frac{\partial p(\boldsymbol{x}_2)}{\partial \boldsymbol{x}_2}\right)\cdot\frac{\boldsymbol{r}}{r}, \tag{6}$$

$$\delta_r(\nabla^2 U) = \left(\nabla^2_{\boldsymbol{x}_1}\boldsymbol{u}(\boldsymbol{x}_1) - \nabla^2_{\boldsymbol{x}_2}\boldsymbol{u}(\boldsymbol{x}_2)\right)\cdot\frac{\boldsymbol{r}}{r}. \tag{7}$$

Gotoh considered the pressure term from the view point of the conditional average of the longitudinal increments of the pressure gradients $\left\langle\delta\nabla_\parallel p|U,r\right\rangle$ for given value of U (see later sections for more detail)[21]. The conditional average is expressed in terms of the quadratic function of U and the coefficients are fixed by some statistical constraints. The qualitative feature of the conditional average was compared with DNS data and the results was found to be satisfactory.

Yakhot studied the equations of the structure functions by using the generating functions which had been

introduced by Polyakov[15, 18]. The same equations were obtained also by Hill by directly dealing with the velocity increments instead of the generating function[20]. In the equations, the terms arising from the convective term of the Navier Stokes equation are expressed in terms of $S_{p,q}$ alone, but the equations again contain contributions from the pressure and viscous terms. Yakhot argued that in the equations of $S_{2n,0}$ the contribution from the viscous term vanishes and the pressure term can be neglected, and concluded $\zeta_{2n,0} = \zeta_{2n-2,2}$. For the equations of $S_{p,2m}$, the pressure term is modelled as the conditional average of the pressure gradient increments for given values of U and V, $\langle \delta\nabla_\perp p | U, V, r\rangle$, which is assumed to be given by the expansion in velocity increments and more general than Gotoh's case. He retained terms up to the second order, and argued a resemblance to "mean-field approximation" in the critical phenomena[18]. This model for the pressure term enabled him to close the equation for the probability density function (PDF) of V in the inverse cascading range in the two dimensional turbulence where the dissipation term is irrelevant. The PDF is found to be Gaussian and the normal scaling for $S_{0,q}$ was obtained. The same closure for the pressure term and a model for the dissipation term yield a closed equation for the PDF of V in three dimensional turbulence, from which the peak value and the asymptotic tail of the PDF are derived, and the scaling exponent for $S_{0,q}$ is computed.

These facts suggest that when the dissipation contributions to the equation of the velocity structure functions are irrelevant the pressure-velocity correlation is a key to understand the anomalous scaling exponents. The average of the pressure term conditioned upon the velocity increment is a kind of statistical projection of the pressure force on the dynamics of the velocity increment and study of its functional form would yield important knowledge of the scaling of the structure functions[18, 20].

Kurien and Sureenivasan studied experimentally the equations of the above structure functions by using the turbulence data measured in the atmospheric boundary layer at the Talyor microscale Reynolds number $R_\lambda \approx 10700$. The width of the inertial range is about one decade. Since the direct measurement of the contributions of the pressure and viscous terms are difficult, they considered first the equations of the structure functions without the pressure and viscous terms, and then examined the balance among the terms in the equations. They concluded that the pressure effects are less important for $S_{2n,0}$ at low order[24]. Further, in order to assess the contributions of the pressure term, they applied Yakhot's theory to the pressure term. With this model, the correlation between the velocity increments and the pressure gradient increment is expressed in terms of the velocity structure functions alone, meaning a closure. They concluded that the pressure and viscous terms are small compared to the other terms consisting of $S_{p,q}$. Also the scaling exponents of $S_{0,q}$, the peak values of the PDF and the asymptotic tail of the PDF of V were examined and compared to Yakhot's prediction. An agreement was observed for these quantities, but insufficient for the pressure contributions. Since there is small but finite degree of anisotropy at second and third order structure functions, and the data in the experiment suffers from noise, the conclusion is very suggestive but inconclusive. Moreover the pressure and viscous terms are examined indirectly through the assumed model to the pressure term. Certainly we are left with lack of precise knowledge for the equation and further study of the equation of $S_{p,q}$ is highly needed. Especillay the direct examination of the pressure and viscous terms is required.

The above story motivates us to examine the dynamical relations for $S_{m,n}(r)$ in terms of DNS. DNS can provide precise data of each term in the equations so that the contributions of the pressure and viscous terms are directly examined. When compared to the atmospheric boundary layer data, $R_\lambda = 460$ of the DNS is low, but degree of isotropy is so well that about one decade of the inertial range width is achieved[26]. In the following, the analysis of the DNS data shows that the pressure term is important to the dynamical equations of the longitudinal correlation functions. Conditional average of the pressure gradient increments is also analysed and its role in the dynamics is discussed.

DYNAMICAL RELATIONS

The scalewise energy budget is well described by the Kárman-Howarth-Kolmogorov equation

$$\frac{4}{5}\bar{\epsilon} r = -S_{3,0} + 6\nu \frac{dS_{2,0}}{dr} + Z_{force}(r), \tag{8}$$

where $Z_{force}(r)$ is an energy input by an external force. Note that this equation does not contain the pressure term. When the external force has a support compact spectrum at low wavenumbers, both the viscous and external force terms can be neglected so that we have Kolmogorov's 4/5 law.

The dynamical relations at higher order in the inertial range are derived from the Navier-Stokes equation. This can be efficiently done by introducing the generating function $Z = \langle \exp(\lambda_\alpha w_\alpha)\rangle$ where $\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{x}_1) - \boldsymbol{u}(\boldsymbol{x}_2)$ is the velocity difference [15, 18]. The resulting equation is

$$\begin{aligned}\frac{dS_{p,q}}{dr} + \frac{d-1+q}{r} S_{p,q} - \frac{(p-1)(q-1+d)}{r(q+1)} S_{p-2,q+2} \\ = -(p-1)\big\langle (\delta\nabla_\parallel p) U^{p-2} V^q \big\rangle \\ -q\big\langle (\delta\nabla_\perp p) U^{p-1} V^{q-1} \big\rangle - D_{p,q},\end{aligned} \tag{9}$$

where d is the dimensionality. $D_{p,q}$ denotes the contri-

butions from the viscous terms

$$D_{p,q} = \nu\Big((p-1)(p-2)\big\langle(\nabla_{\boldsymbol{X}}U)^2U^{p-3}V^q\big\rangle$$
$$+2q(p-1)\big\langle(\nabla_{\boldsymbol{X}}U)\cdot(\nabla_{\boldsymbol{X}}V)U^{p-2}V^{q-1}\big\rangle$$
$$+q(q-1)\big\langle(\nabla_{\boldsymbol{X}}V)^2U^{p-1}V^{q-2}\big\rangle\Big), \qquad (10)$$

where $\boldsymbol{X} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$, $\nabla_{\boldsymbol{X}}U = \partial u_l(\boldsymbol{x}_1)/\partial\boldsymbol{x}_1 - \partial u_l(\boldsymbol{x}_2)/\partial\boldsymbol{x}_2$, $\nabla_{\boldsymbol{X}}V = \partial u_t(\boldsymbol{x}_1)/\partial\boldsymbol{x}_1 - \partial u_t(\boldsymbol{x}_2)/\partial\boldsymbol{x}_2$, and u_l and u_t are the longitudinal and transverse components of the velocity vector, respectively. The external force term is also neglected[18, 20], and the pressure terms appear in Eq.(9).

In the inertial range, Yakhot, Hill and Boratav, Hill, and Kurien and Sreenivasan have discussed Eq. (9) for some set of (p, q) without pressure and viscous contributions [18–20, 24]. Following their discussion and for latter convenience we write a first few series of Eqs.(9) without the pressure and viscous terms as

$$\frac{dS_{4,0}}{dr} + \frac{2}{r}S_{4,0} = \frac{6}{r}S_{2,2}, \qquad (11)$$

$$\frac{dS_{6,0}}{dr} + \frac{2}{r}S_{6,0} = \frac{10}{r}S_{4,2}, \qquad (12)$$

$$\frac{dS_{8,0}}{dr} + \frac{2}{r}S_{8,0} = \frac{14}{r}S_{6,2}, \qquad (13)$$

for the longitudinal structure functions and

$$\frac{dS_{2,2}}{dr} + \frac{4}{r}S_{2,2} = \frac{4}{3r}S_{0,4}, \qquad (14)$$

$$\frac{dS_{2,4}}{dr} + \frac{6}{r}S_{2,4} = \frac{6}{5r}S_{0,6}, \qquad (15)$$

$$\frac{dS_{4,2}}{dr} + \frac{4}{r}S_{4,2} = \frac{4}{r}S_{2,4}, \qquad (16)$$

for the mixed ones[20]. If the pressure and viscous terms can be neglected in the inertial range, we obtain the relations between the scaling exponents of the longitudinal, mixed, and transverse structure functions as $\zeta_{2n,0} = \zeta_{2n-2,2}$, $\zeta_{2,2n} = \zeta_{0,2n+2}$, and so on. In the following we examine the above relations quantitatively in terms of the DNS data.

EXAMINATION WITH DNS DATA

The turbulence in a periodic cubic box with size of 2π is simulated by using the Fourier spectral method for space, and Runge Kutta Gill method for time advancing. The Gaussian, white-in-time random solenoidal force is applied to the first few shells in the wavevector space to maintain statistically steady state. The total number of grid points is $N = 1024^3$ and $k_{max}\eta = 0.96$, where $\eta = (\nu^3/\bar{\epsilon})^{1/4}$ is the Kolmogorov scale. The value 0.96 is slightly smaller than unity, but this does not affect the statistics of the inertial range. The ensemble average is taken over time average with the period of about four large eddy turnover time. Detail of the simulation is found in Gotoh et al.[22, 26].

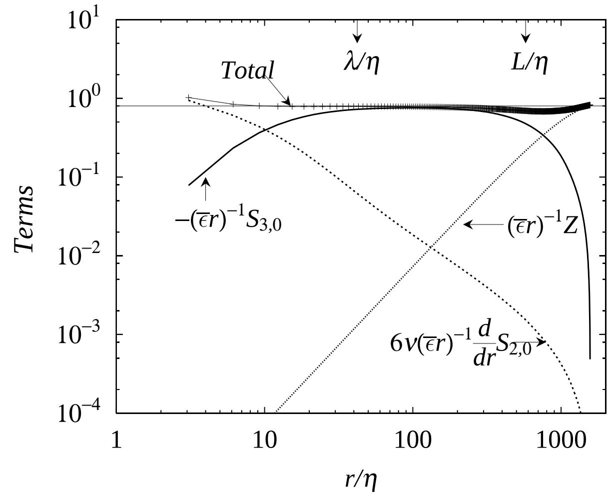

Fig.1 4/5-law at $R_\lambda = 460$[26].

Figure 1 shows each term of the Kárman-Howarth-Kolmogorov equation (8) divided by $\bar{\epsilon}r$. When r decreases, the forcing term decreases and $-S_{3,0}(r)$ quickly rises to the 4/5 level and decays in the dissipation range. The energy budget follows Kolmogorov's 1941 theory and the inertial range certainly exists.

In Figs. 2, 3 and 4, both sides of Eqs. (11), (12) and (13) are plotted, respectively, and the ratio of the difference of both sides to the left hand side ($|l.h.s. - r.h.s|/|l.h.s.|$) is also shown in the figures. If the contribution from the pressure gradient is negligible, both curves must collapse at least over a part or whole of the inertial range. Figures show that there exist differences and the left hand side of Eq. (11) in Fig.2 is bigger by factor 1.2 than the right hand side of Eq. (11). Distance between the two curves is almost constant over the inertial range, but increases with the order n. Corresponding to this, the ratio of the difference between both sides to the left hand side is nearly constant over the inertial range, which suggests that the contribution from the pressure gradient term scales almost in the same way as $S_{2n,0}$.

When the pressure term is included in the right hand side of Eqs. (11), (12) and (13), the curve of the right hand side (dash dotted line with plus symbol) perfectly collapses with the curve of the left hand side, which means that the pressure gradient term has the same order of contributions as the mixed correlation term. We have examined the contributions from the dissipation term and found that they are very small compared to the other terms of Eq. (9) with $p = 2m$ and $q = 0$, consistent with the theoretical prediction[18]. For the dynamical equations of the odd order moments such as $S_{2n+1,0}(r)$, all the terms would contribute equally but unfortunately we

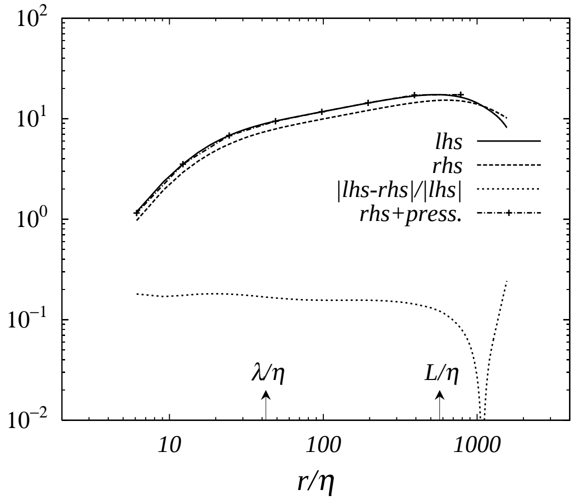

Fig.2 $\frac{\partial S_{4,0}}{\partial r} + \frac{2}{r} S_{4,0} = \frac{6}{r} S_{2,2} \quad (-3\langle \delta_r p_x U^2 \rangle)$.

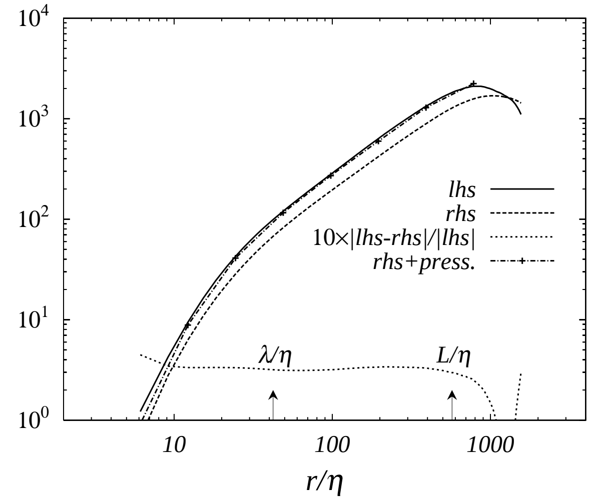

Fig.4 $\frac{\partial S_{8,0}}{\partial r} + \frac{2}{r} S_{8,0} = \frac{14}{r} S_{6,2} \quad (-7\langle \delta_r p_x U^6 \rangle)$.

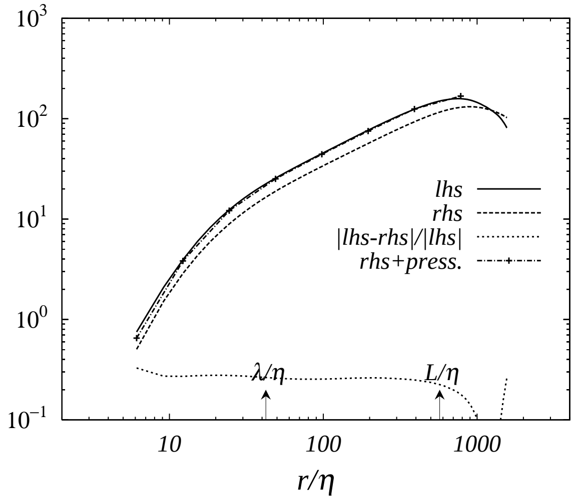

Fig.3 $\frac{\partial S_{6,0}}{\partial r} + \frac{2}{r} S_{6,0} = \frac{10}{r} S_{4,2} \quad (-5\langle \delta_r p_x U^4 \rangle)$.

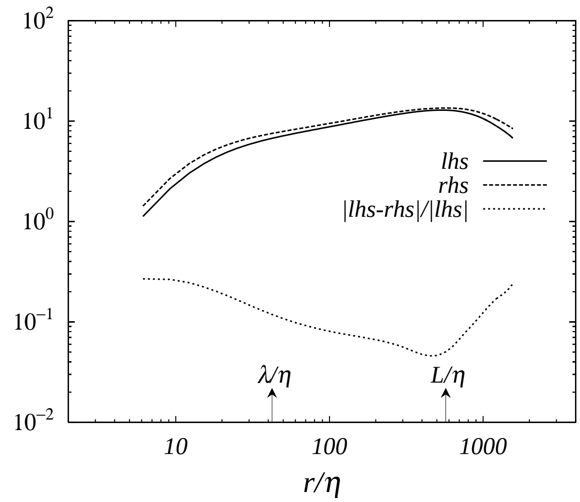

Fig.5 $\frac{\partial S_{2,2}}{\partial r} + \frac{4}{r} S_{2,2} \approx \frac{4}{3r} S_{0,4}$

have not computed these terms.

Figures 5, 6 and 7 show the both sides of Eqs. (14), (15) and (16). The curves for both sides of these equations are very close to each other for whole or part of the separation below the integral scale. Although it is not clear whether the difference between the curves remains to be small when the Reynolds number is increased, the degree of contribution of the pressure gradient to the dynamical equations in these cases is weaker than the case of the longitudinal case.

These observations suggest that as far as the dynamics of the longitudinal velocity increments is concerned, the pressure term is very essential, while it is secondary for the dynamics of the transverse velocity increments. If the pressure term is absent, sharp shock is formed as in the Burgers turbulence, but in reality the pressure suppresses the formation of shock. This anti-shock action of the pressure would become stronger when $|U|$ with negative sign becomes larger. This implies that the larger the longitudinal velocity increments is, the more significant the contribution of the pressure term becomes (see more discussion in Sec.4). On the other hand, near the shock region, transverse velocity increments are closely tied to shearing motion as results of shock killer action of the pressure. Both positive and negative transverse velocity increment occur with equal probability so that the correlation between the transverse velocity increments and pressure gradient would have a good chance to be small.

When the pressure term is included in the dynamical equation Eq.(9), it is not clear whether or not the scaling exponents $\zeta_{2n,0}$ and $\zeta_{2n-2,2}$ are equal. We need precise knowledge about the correlation between the increments of the pressure gradient and longitudinal velocity. In the following we examine the pressure term from the view point of the conditional average.

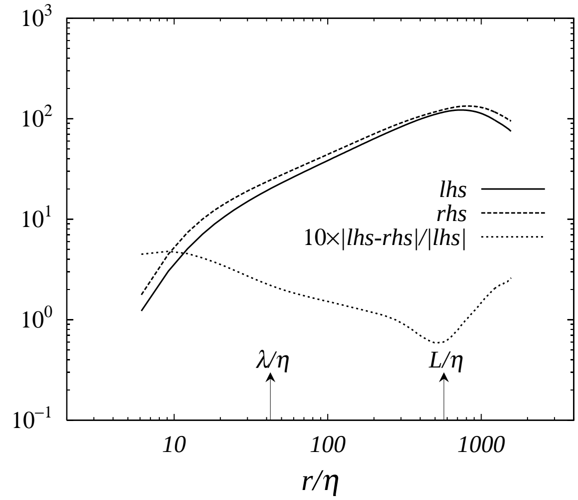

Fig.6 $\frac{\partial S_{2,4}}{\partial r} + \frac{6}{r} S_{2,4} \approx \frac{6}{5r} S_{0,6}$

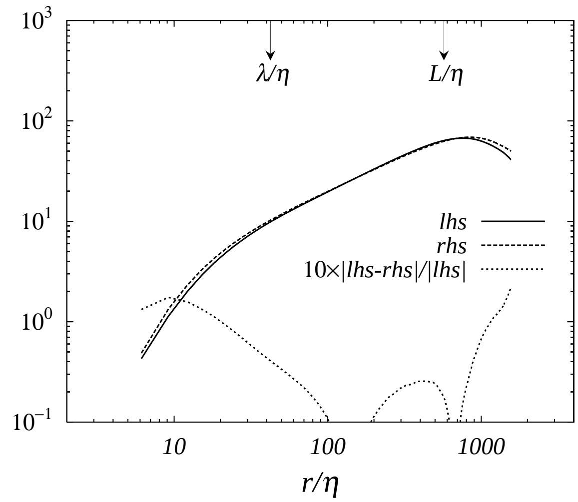

Fig.7 $\frac{\partial S_{4,2}}{\partial r} + \frac{4}{r} S_{4,2} \approx \frac{4}{r} S_{2,4}$

CONDITIONAL AVERAGE OF THE PRESSURE GRADIENT INCREMENT

The correlation function $\left\langle (\delta p_{,x}) U^{2n-2} \right\rangle$ can be written as follows

$$\left\langle (\delta p_{,x}) U^{2n-2} \right\rangle = \int_{-\infty}^{\infty} \langle \delta p_{,x} | U, r \rangle U^{2n-2} P(U, r) dU, \quad (17)$$

where $\langle \delta p_{,x} | U, r \rangle$ is the conditional average of the pressure gradient increment for a fixed value of U and $P(U, r)$ is the PDF of U with separation r. If $\langle \delta p_{,x} | U, r \rangle$ is given as a polynomial function of U, then $\left\langle (\delta p_{,x}) U^{2n-2} \right\rangle$ is expressed in terms of the moments of U. For the moment we assume that $\langle \delta p_{,x} | U, r \rangle$ is a quadratic function of U. Underlying motivation for this is that (1) the source term of the Poisson equation for the pressure is quadratic in the velocity gradient, i.e., when $u' = \gamma u$ we have $p' = \gamma^2 p$, and (2) for large U the local Reynolds number is quite large so that the fluid motion would follow Bernoulli's theorem locally, implying that the pressure field is proportional to the square of the velocity. This is sketched as follows.

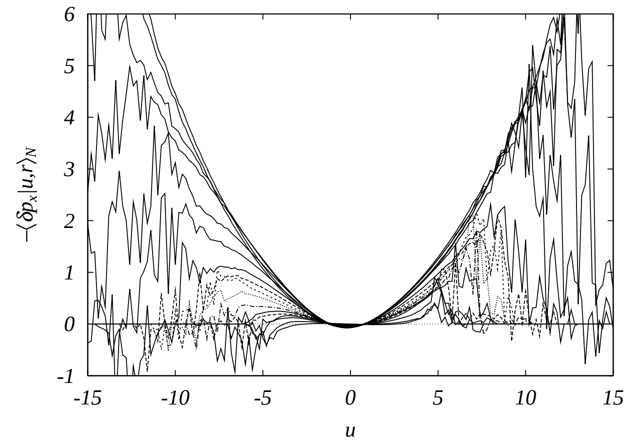

Fig.8 $-\langle \delta p_x | u, r \rangle_N$ for various r/η ($r_{min}/\eta = 3$ and $r_{max}/\eta = 1565$.). As the separation r increases the curvature becomes smaller.

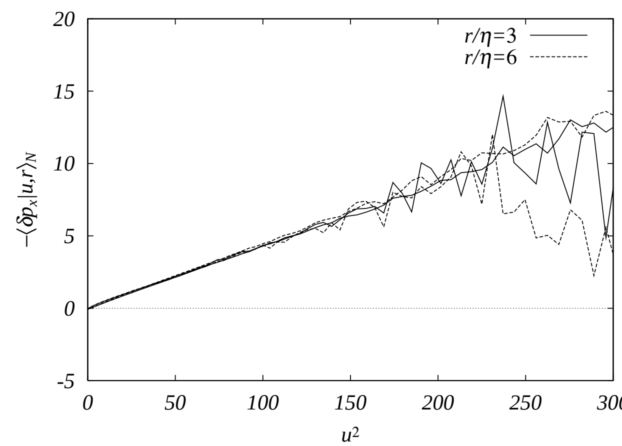

Fig.9 $-\langle \delta p_x | u, r \rangle_N$ plotted against u^2 for $r/\eta = 3$ and 6.

Consider a vortex tube and a cylindrical surface surrounding the tube. On this surface, the velocity vector is nearly along the circumferential direction and the vorticity vector is approximately parallel to the axis of the vortex tube. Then the vector $\boldsymbol{u} \times \boldsymbol{\omega}$ is parpendicular to the surface, the Bernoulli surface[23]. On this surface, the Bernoulli theorem is $p(\boldsymbol{x}_1) + \boldsymbol{u}^2(\boldsymbol{x}_1)/2 = H_1$ where $\rho = 1$ is assumed for simplicity and H_1 is a Bernoulli constant on the surface. Take the gradient, we have $-\nabla_1 p_1 = \boldsymbol{u}_1 \cdot \nabla_1 \boldsymbol{u}_1$. Then subtract from this equation the same expression evaluated at $\boldsymbol{x}_2$ on a different Bernoulli surface, and introduce the new cooridnate as $\boldsymbol{X} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ and $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, we obtain

$$\begin{aligned} -\delta \nabla p(\boldsymbol{X}, \boldsymbol{r}) = & \, \boldsymbol{V}(\boldsymbol{X}, \boldsymbol{r}) \cdot \nabla_{\boldsymbol{X}} \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r}) \\ & + \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r}) \cdot \nabla_{\boldsymbol{r}} \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r}), \end{aligned} \quad (18)$$

where $\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{r}) = (\boldsymbol{u}(\boldsymbol{x}_1) + \boldsymbol{u}(\boldsymbol{x}_2))/2$ which would be a slowly varing function of $\boldsymbol{X}$ and $\boldsymbol{r}$ compared to $\delta \boldsymbol{u}$. If

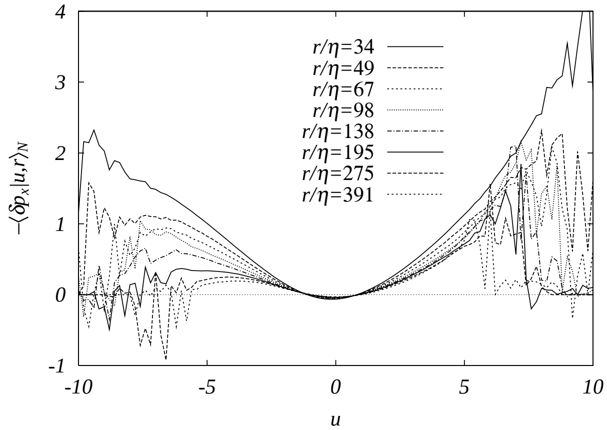

Fig.10 $-\langle \delta p_x | u, r\rangle_N$ for r/η in the inertial range.

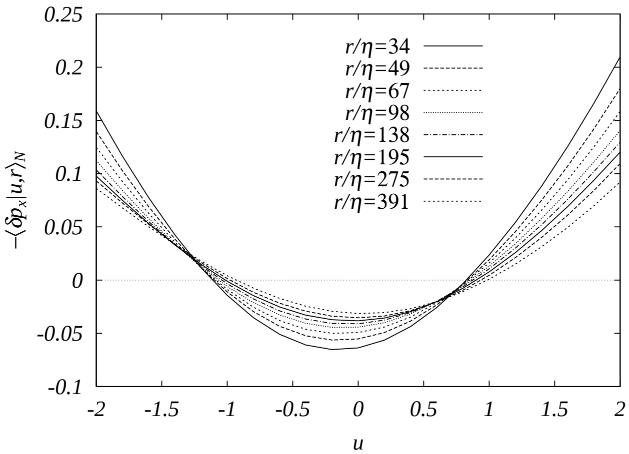

Fig.11 Close up of $-\langle \delta p_x | u, r\rangle_N$ for r/η in the inertial range.

we expect $\nabla_{\boldsymbol{X}} \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r}) \propto \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r})$ and $\nabla_{\boldsymbol{r}} \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r}) \propto \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{r})$, Eq.(18) becomes quadratic in $\delta \boldsymbol{u}$. Moreover, since the Bernoulli constant differs on each Bernoulli surface, this fact would introduce an additional factor $\delta \boldsymbol{H}$, independent of the velocity difference, to Eq.(18). The final expression would be $-\delta \nabla p = \boldsymbol{A}_0 + \boldsymbol{A}_1 \delta \boldsymbol{u} + \boldsymbol{A}_2 \delta \boldsymbol{u} \delta \boldsymbol{u}$, where $\boldsymbol{A}_i$'s are tensor functions of $\boldsymbol{X}$ and $\boldsymbol{r}$. The average over $\boldsymbol{X}$ or statistical ensemble conditional upon the velocity increments would yield the quadratic function of $\delta \boldsymbol{u}$ for $-\langle \delta \nabla p | \delta \boldsymbol{u}, \boldsymbol{r}\rangle$. The above argument is very rough and needs critical examination.

The DNS data of the conditional average at $R_\lambda = 460$ is shown in Fig.8 for the separation from the dissipation to the integral scale. In the plot, u and $\langle \delta p_{,x} | u, r\rangle_N$ are normalized by their standard deviations, respectively. The curves are very close to the quadratic function of u. As the separation increases the curvature decreases, and the curves peel off the quadratic form at large amplitude of u. We infer that this is due to the fact that the number of large amplitude events becomes smaller as the amplitude u increases.

To see how close the curve is to the quadratic function

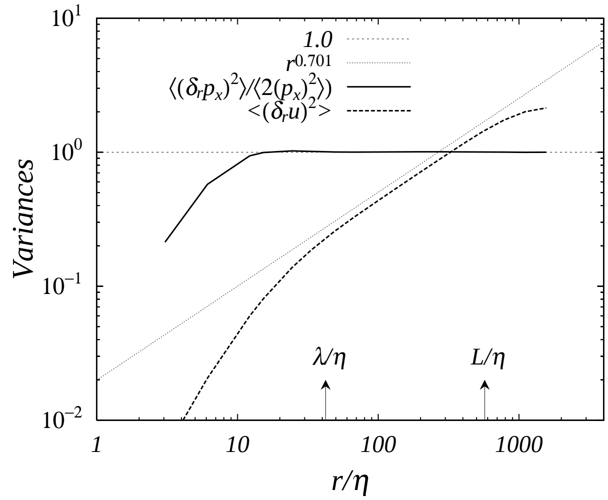

Fig.12 Variation of the normalized variance of the pressure gradient increment $\frac{3}{2} F_{\nabla p}^{-1} \bar{\epsilon}^{-3/2} \nu^{1/2} \langle (\delta p_x)^2 \rangle$ with separation.

of u, $\langle \delta p_{,x} | u, r\rangle_N$ at $r/\eta = 3$ and 6 is plotted against u^2 for both positive and negative u in Fig. 9. Curves are straight up to about $u^2 \approx 200$, strongly suggesting that the curves are quadratic funtions of u. The curves for the inertial range separation are plotted in Fig.10. For small value of u, $\langle \delta p_{,x} | u, r\rangle_N$ is negative and positive at large u. Close inspection of Fig.11 reveals that the vertex of the quadratic function locates at negative u and moves towards the origin as the separation increases.

It is very interesting to see how the quadratic function in U for $\langle \delta p_{,x} | U, r\rangle$ is fixed. The assumed quadratic form is

$$-\langle \delta p_{,x} | U, r\rangle = a_2(r) U^2 + a_1(r) U + a_0(r). \tag{19}$$

From the condition $\langle \delta p_{,x}\rangle = 0$, we have $a_0 = -a_2 S_{2,0}(r)$ where we used $\langle U\rangle = 0$ and $\langle U^2\rangle = S_{2,0}(r)$. Also the condition $\langle U \delta p_{,x}\rangle = 0$ by the homogeneity, isotropy and incompressibility yields $a_1(r)\langle U^2\rangle = -\langle U^3\rangle a_2(r)$[11, 25]. Combining these two relations we obtain $\langle \delta p_{,x} | u, r\rangle_N$ in non-dimensional form as

$$\begin{aligned} -&\langle \delta p_{,x} | u, r\rangle_N \\ &= -\left\langle (\delta p_{,x})^2 \right\rangle^{-1/2} \langle \delta p_{,x} | U, r\rangle \\ &= \frac{\hat{a}_2(r)}{\left\langle (\delta p_{,x})^2 \right\rangle^{1/2}} \frac{S_{2,0}(r)}{r} \left[u^2 - s_3(r) u - 1 \right], \end{aligned} \tag{20}$$

where $\hat{a}_2(r) = r a_2(r) > 0$, and $s_3(r) = S_{3,0}(r)/[S_{2,0}(r)]^{3/2}$ is the skewness factor. Since $S_{3,0}(r) = -(4/5)\bar{\epsilon} r$ in the inertial range, we obtain

$$s_3(r) = -\frac{4}{5} C_0^{-3/2} \left(\frac{r}{L}\right)^{1 - \frac{3}{2}\zeta_{2,0}}, \tag{21}$$

where $S_{2,0}(r) = C_0 \bar{\epsilon}^{2/3} r^{2/3} (r/L)^{\zeta_{2,0} - 2/3}$ is used, and L is a macro length scale, $C_0 = \frac{27}{55}\Gamma(1/3) K \approx 1.32 K$, and K is the Kolmogorov constant. We expect that $\hat{a}_2(r)$ tends

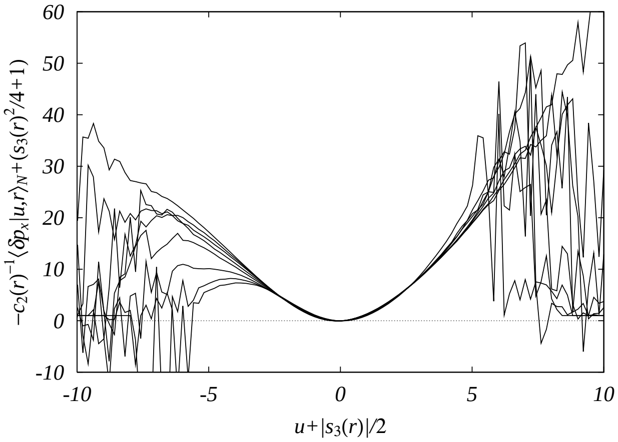

Fig.13 Scaled $-\langle\delta p_x|u,r\rangle_N$.

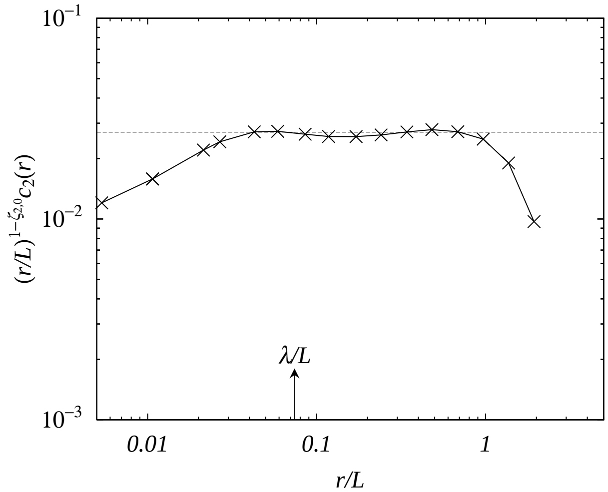

Fig.14 Compensated plot of the coefficients $c_2(r)$.

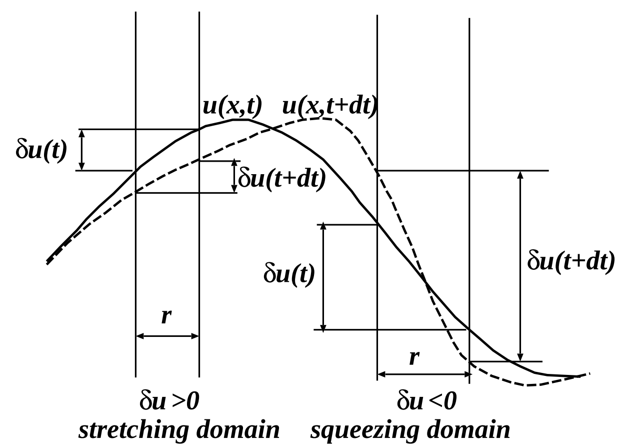

Fig.15 The role of the pressure term on the dynamics of the longitudinal velocity increment. The convection stretches the fluid portion with $\delta U > 0$ ($\xi = \partial U/\partial x$ decreases) and squeeze the fluid portion with $\delta U < 0$ ($|\xi|$ increases). The modeled pressure gradient gives positive contribution $c_2 U^2$ to $\partial U/\partial t$, that is, $\xi(>0)$ of the stretched portion is increased and $|\xi|(\xi<0)$ of the squeezed one is decreased. This means that the stretched and squeezed portions are pushed back left by the pressure gradient.

to a positive universal constant in the inertial range. The variance $\left\langle(\delta p_{,x})^2\right\rangle = 2(\left\langle p_{,x}^2\right\rangle - \langle p_{,x}(\boldsymbol{x}+r\boldsymbol{e}_1)p_{,x}(\boldsymbol{x})\rangle)$ quickly approaches the value $2\left\langle p_{,x}^2\right\rangle$ when r becomes larger than the dissipation scale as seen in Fig.12, so that it is well approximated as

$$\left\langle(\delta p_{,x})^2\right\rangle = \frac{2}{3}\bar{\epsilon}^{3/2}\nu^{-1/2}F_{\nabla p}(R_\lambda), \tag{22}$$

where $F_{\nabla p}$ is a nondimensional constant defined by $\left\langle(\nabla p)^2\right\rangle = \bar{\epsilon}^{3/2}\nu^{-1/2}F_{\nabla p}$. $F_{\nabla p}(R_\lambda)$ is R_λ dependent for low to moderate R_λ and slowly tends to a constant[22, 27]. Using these quantities we write the conditional average as

$$\begin{aligned}
-&\langle\delta p_{,x}|u,r\rangle_N \\
&= c_2(r)\left[\left(u+\frac{|s_3(r)|}{2}\right)^2 - \left(\frac{|s_3(r)|^2}{4}+1\right)\right] \\
&= \chi\left(\frac{r}{L}\right)^{\zeta_{2,0}-1}\left[\left(u+\frac{|s_3(r)|}{2}\right)^2 - \left(\frac{|s_3(r)|^2}{4}+1\right)\right],
\end{aligned} \tag{23}$$

in the inertial range, where

$$\chi = \frac{AC_0}{\sqrt{(2/3)F_{\nabla p}(R_\lambda)}}R_\lambda^{-1/2}, \tag{24}$$

and A is a nondimensional constant. Figure 13 shows the scaled plot of $-c_2(r)^{-1}\langle\delta p_{,x}|u,r\rangle_N + \left(\frac{|s_3(r)|^2}{4}+1\right)$ against $u+\frac{|s_3(r)|}{2}$ for the inertial range separation. Collapse of curves for low to moderate amplitude is excellent.

We have computed the curvature of $-\langle\delta p_{,x}|u\rangle_N$ at the origin and plotted in Fig.14. Since $c_2(r)$ is assumed to obey the power law $c_2(r) = \chi(r/L)^{-\alpha}$ in the inertial range, the curve is multiplied by $(r/L)^{1-\zeta_{2,0}}$ with $\zeta_{2,0} = 0.701$ (see Fig.14) [26]. The nondimensional constant χ and the exponent α were found to be 0.027 and 0.267, respectively, by the least square fit in the range of $0.09 \leq r/L \leq 0.3$. The latter value 0.267 is slightly smaller than the value $1-\zeta_{2,0} = 1-0.701 = 0.299$.

From the above observation, we conclude that the conditional average $-\langle\delta p_{,x}|u,r\rangle_N$ is well approximated by the quadratic function of u with the curvature which decreases with the separation. The quadratic function of u for $-\langle\delta p_{,x}|u,r\rangle_N$ is consistent with the anti-shock action of the pressure. Locally the equation for U may be described by the Burgers type equation with the pressure effect:

$$\frac{DU}{Dt} \equiv \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} = -\frac{\partial \delta p}{\partial X} + 2\nu\frac{\partial^2 U}{\partial r^2} + K(X,r,t), \tag{25}$$

where X is a x-coordinate moving with the average velocity at two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and K denotes the remaining terms arising from the Naveir-Stokes equation[14]. If the pressure gradient term is modeled by the quadratic function of U as inferred from the conditional average $-\langle \delta p_{,x}|U,r\rangle$, all the terms except K term are expressed in terms of U. This replacement is thought to be a kind of statistical projection of the pressure gradient onto U-space. The K may be modeled as random force, but it is irrelevant in the following discussion.

To make discussion simpler, we take the leading term of $-\langle \delta p_{,x}|U,r\rangle \approx a_2(r)U^2, a_2 > 0$. In the large negative U region (squeezing domain), the convective term produces a shock as in Fig.15. The pressure gradient term makes a positive contribution a_2U^2 to $\partial U/\partial t$, which means that large (negative) amplitude $U(t+dt)$ is decreased. On the other hand, for large positive U regions, those regions are stretched by the convective action so that the positive amplitude decreases[12, 13]. When the pressure gradient term is added, again this gives positive contributions to $\partial U/\partial t$, thus $U(t+dt)$ increases. Therefore the pressure gradient term acts in such a way that it suppresses the formation of singularity such as shocks by the convective motion, and this can be understood as the action which the curve $U(t+dt)$ is pushed back toward left by the pressure gradient in Fig.15. This implies that the pressure term is an indispensable ingredient for the dynamics of the longitudinal velocity increment.

IMPLICATION OF THE PRESSURE GRADIENT TERM

When the closure (19) is included in Eq.(9) with $p = 2n$ and $q = 0$, the pressure-velocity correlation term becomes

$$-\big\langle (\delta p_{,x})U^{2n-2}\big\rangle = \frac{\hat{a}_2(r)}{r}\Bigg(S_{2n,0}(r) - \frac{S_{3,0}(r)}{S_{2,0}(r)}S_{2n-1,0}(r) - S_{2,0}(r)S_{2n-2,0}(r)\Bigg). \qquad (26)$$

Suppose that when the Ryenolds number and n are very large, $S_{2n,0}(r)$ in Eq.(26) is dominant in the inertial range. This requires $\zeta_{2n,0} \le \zeta_{2n-1,0} + 1 - \zeta_{2,0}$. On the other hand, the realizability condition requires that $\zeta_{p,0}$ is a nondecreasing function of p, meaning $\zeta_{2n-1,0} \le \zeta_{2n,0}$. Both constraints suggests that $0 \le \zeta_{2n,0} - \zeta_{2n-1,0} \le 1 - \zeta_{2,0} \approx 0.3$. In K41, $\zeta_{2n,0} - \zeta_{2n-1,0} = 1/3$. From the DNS data, the above requirement is satisfied about at the order greater than four.

Dominance of $S_{2n,0}$ over other two terms of Eq.(26) means that Eq.(9) becomes

$$\frac{dS_{2n,0}}{dr} + \frac{2}{r}S_{2n,0} = \frac{2(2n-1)}{r}S_{2n-2,2} + (2n-1)\frac{\hat{a}_2(r)}{r}S_{2n,0}, \qquad (27)$$

asymptotically in the inertial range. Suppose that $\hat{a}_2(r) \propto r^\gamma$ in the inertial range. When $\gamma < 0$, since the two terms of the right hand side of Eq.(27) are positive and the last term becomes dominant, the both side can not balance. This means that the case of $\gamma < 0$ can not occur. When $\gamma = 0$, both sides of Eq.(27) balances, thus we have $\zeta_{2n,0} = \zeta_{2n-2,2}$. If $\gamma > 0$, the last term of the right hand side of Eq.(27) becomes less important so that the left hand side balances with the first term of the right hand side and $\zeta_{2n,0} = \zeta_{2n-2,2}$. The conclusion of the above arguments is $\zeta_{2n,0} = \zeta_{2n-2,2}$ for large n, but this equality is totally dependent on the assumption of the quadratic function in U for the conditional average (see also the discussion by Hill[20]).

Although the DNS data shows that $\hat{a}_2(r) = \chi C_0^{-1}(2F_{\nabla p}/3)^{1/2}$ is approximately constant, there is small but finite difference between the exponent $\alpha = 0.267$ and $1 - \zeta_{2,0} = 0.299$. If we regard these two values correct, we have $\gamma = 0.299 - 0.267 = 0.032 > 0$. However, we think that our data is not accurate enough for definite judgement, and the preceding argument about the role of the pressure term suggests that $\gamma = 0$ is very plausible.

It is important and interesting to seek physical explanation for possible r-dependence of $\hat{a}_2(r)$, if it is the case. To the authors' best knowledge, no theory has been developed to compute $\hat{a}_2(r)$ including a nondimensional constant. We present one possible argument in the following.

The pressure is given by the spatial integral of the source term which is quadratic in the velocity gradients, so that the pressure gradient is given by

$$\nabla p = -\int_V \nabla_{\boldsymbol{x}} G(\boldsymbol{x}-\boldsymbol{y})B(\boldsymbol{y})d\boldsymbol{y}, \qquad (28)$$

$$G(\boldsymbol{x}-\boldsymbol{y}) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}, \qquad (29)$$

for the periodic boundary condition in the present study, where $B = \nabla\boldsymbol{u} : \nabla\boldsymbol{u}$. The integral is roughly evaluated as the product of B, ∇G and an effective volume V_{effect}[28]. The derivative of the Poisson kernel ∇G does not yield non-integer power, and the source term B would contribute to $S_{2n,0}$. The effective volume is the one over which the source term has a dominant contribution to the integral, and would change its shape and size depending on the amplitude and its spatial coherency of B. This effective volume is characterized by a geometrical non-integer dimension. Therefore it is plausible that the effective volume V_{effect} may induce a non-integer power law dependence of $\hat{a}_2(r)$ on the scale r.

SUMMARY

We have examined the dynamical relations for the various kinds of the velocity structure functions in terms of

the DNS data. It was argued that the pressure term is a key to understand the scaling of the velocity structure functions. The contribution of the pressure gradient is important to the dynamics of the longitudinal structure functions, while in the case of the transverse or mixed ones the pressure gradient may or may not be important.

To assess the statistical role of the pressure gradient term in the dynamics of the longitudinal velocity increment, we have computed the conditional average of the pressure gradient increment with a given value of U and r. It was found that the quadratic function in U is a good approximation to the conditional average. By applying two constraints to the conditional average and using Kolmogorov's 4/5 law, we have theoretically determined two coefficients of the quadratic function but there remains one undetermined coefficients.

The role of the pressure gradient was argued in terms of the equation for U which is similar to the Burgers' equation but with the term which models the pressure gradient. The pressure gradient acts to resist to stretching and squeezing of the fluid element, leading to shock smearing, which in turn means that the scaling exponent of $S_{2n,0}$ is not constant in n, unlike the Burgers case, and is a slowly increasing function of the order. This means that the pressure is an intermittency killer (Kraichnan 1991)[29].

The quadratic function form for the conditional average of the pressure gradient increment leads to the closure of the moment equation for $S_{2n,0}$. If higher order polynomial form is used, the closure fails. However, if a thin vortex tube is the most singular object under the Navier-Stokes dynamics, the argument using the Bernoulli theorem nearby the tube described in Sec.4 suggests that there are no higher order terms than the second order. The bottom line is $\zeta_{2n,0} = \zeta_{2n-2,2}$. It was argued that the effective volume is an important factor for the dependence of $\hat{a}_2(r)$ on r. Further theoretical study is necessary. The conditional average of the pressure gradient also leads to the closure of the Liouville equation for the PDF of U. This analysis is now underway and will be reported elsewhere.

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